818. Determine the solution h of the following equation (originating from acoustics)

$$h(t) = 1 - \int_{-\infty}^{t} \sum_{n=1}^{\infty} a^n h(\tau - n) d\tau \quad \text{for } t \geqslant 0$$

$$h(t) = 0 \quad \text{for } t < 0.$$

(S.. RIENSTRA)

Solutions by P.J. VAN ALBADA, J. BOERSMA, A.A. JAGERS, S.W. RIENSTRA.

P.J. VAN ALBADA and S.W. RIENSTRA verify the right formula for h after a clever guess. The other two authors use Laplace transform.

SOLUTION by J. BOERSMA.
The equation is solved by Laplace transforms. Introduce

$$H(s) = \mathfrak{L}\{h(t)\} = \int_{0}^{\infty} h(t) e^{-st} dt.$$

Then the equation for $h(t)$ transforms into

$$H(s) = \frac{1}{s} - \sum_{n=1}^{\infty} a^n e^{-ns} \frac{H(s)}{s} = \frac{1}{s} - \frac{ae^{-s}}{1 - ae^{-s}} \frac{H(s)}{s},$$

where the series converges if $\mathrm{Re}(s) > \log|a|$. The solution for $H(s)$ is readily determined and is expanded in a power-series convergent for $\mathrm{Re}(s) > \log|a|$:

$$H(s) = \frac{1 - ae^{-s}}{s - a(s-1)e^{-s}} = \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{s-1}{s}\right)^n [a^n e^{-ns} - a^{n+1} e^{-(n+1)s}].$$

To determine $h(t)$, we apply a term-by-term inverse Laplace transformation using the auxiliary inverse transform

$$\mathfrak{L}^{-1}\left\{\frac{1}{s}\left(\frac{s-1}{s}\right)^n e^{-t_0 s}\right\} = L_n(t - t_0) U(t - t_0), \quad t_0 \geqslant 0;$$

here, U stands for the unit step function defined by $U(x) = 1$ for $x \geqslant 0$, $U(x) = 0$ for $x < 0$ and L_n is the n-th Laguerre polynomial defined by

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} \frac{x^k}{k!},$$

cf. [1, Sec. 5.5.2]. As a result we find

$$h(t) = \sum_{n=0}^{\infty} [a^n L_n(t-n) U(t-n) - a^{n+1} L_n(t-n-1) U(t-n-1)]$$

$$= 1 + \sum_{n=1}^{[t]} a^n [L_n(t-n) - L_{n-1}(t-n)], \; t \geqslant 0.$$

By use of the recurrence relation

$$L_n(x) - L_{n-1}(x) = -\frac{x}{n} L_{n-1}^{(1)}(x),$$

where the (generalized) Laguerre polynomial $L_{n-1}^{(1)}(x)$ is given by

$$L_{n-1}^{(1)}(x) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{n-k-1} \frac{x^k}{k!},$$

the solution for $h(t)$ can also be represented by

$$h(t) = 1 - \sum_{n=1}^{[t]} \frac{a^n}{n} (t-n) L_{n-1}^{(1)}(t-n), \quad t \geqslant 0.$$

The latter form clearly shows that $h(t)$ is continuous at $t = n$, $n \in \mathbb{N}$.

Reference

1. W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and theorems for the special functions of mathematical physics*, 3rd Edition, Springer, Berlin, 1966.